Evaluation of a P/C Insurer's Ultimate Loss Estimates in a Changing Environment

An Honors Thesis (HONRS 499)

by

Paul M. Pleva

Thesis Advisor

Dr. Rebecca L. Pierce

Dr. Rebecca L. Pierce

Ball State University

Muncie, Indiana

April, 1995

Purpose of Thesis

The property and casualty actuary is responsible for developing reasonable estimates of an insurer's ultimate losses. A changing environment can increase the error of these estimates and cause one to question their soundness. The purpose of this thesis is to model and examine estimation strategies under a series of hypothetical environments. This produces a "road map" of the effects various changes in an environment produce on the estimates. A further result is the knowledge of which estimation techniques are robust under these hypothetical environments. This final collection of models and the subsequent analysis will become a part of the actuarial tool box by providing additional information to help property and casualty actuaries estimate ultimate losses.

Table of Contents

1 *Introduction*

The actuary's tool box holds several techniques to calculate reasonable estimates for ultimate losses. Just as a particular size screwdriver fits one screw better than another, some methods of estimation will fit environmental changes better than others. With a better fit, the resulting estimates will be closer to the true ultimate loss of the insurer. Knowledge of a reasonable ultimate loss estimate is important in maintaining an accurate balance sheet (creating a reserve for future liabilities) as well as creating and updating rates. If reserves are understated, a firm will not know it's true financial position, while on the other hand, overstated reserves will reduce the firm's owner's equity. In addition, understated ultimate loss estimates used in the process of rate making will produce rates that are insufficient to cover a firm's losses. Overstated estimates will provide for solvency, but similarly produce excessive rates, inducing insureds to switch to an insurer with lower rates.

Some factors of a changing environment appear to have a dramatic effect on what produces a good estimate. The purpose of this analysis is to create a "road map" of the effects of these changes and the most accurate methods of estimating ultimate losses through the development of a deterministic claims environment model. This model and the subsequent analysis are intended to give the actuary greater insight into various estimation methods and the ability to select factors that yield better estimates.

A basic comprehension of the estimation methods used by the property and casualty insurance industry is required for understanding the ideas discussed within this paper. A brief overview and example follow, which should provide adequate descriptions of the techniques. However, the reader is urged to consult the sources listed in the Bibliography for further information, in particular, the Foundations of Casualty Actuarial Science.

1.1 Methods of Estimation

The primary technique of estimating ultimate losses in the property and casualty industry relies on an analysis of past loss data. Therefore the data must be grouped appropriately into homogeneous categories, such as by the type of coverage or even the geographical location for which the coverage applies. Furthermore, the actuary needs to be able to associate a given loss with an accident period, report period, or policy period. Each period associates claims to a time interval, commonly a length of one year or quarter, where an accident period is the time period during which the accident actually occurred and the report period is the time frame in which the insured reported the claim. A policy period is the time covered by a certain policy, for example, an auto insurance policy may cover a year starting July 1 and ending June 30 of the following calendar year. A claim is attributed to this policy year regardless of the time that the insurer pays, so long as the loss was incurred in this policy year.

The problem with using past data to estimate future losses lies in the fact that for any given period (whether it is accident, report or policy), a certain percentage of the loss may not yet be reported to the insurer. This loss falls into two categories, incurred but not reported claims (IBNR) and allocated expenses. Allocated expenses are the costs that the insurer incurs as a result of adjusting the claim. For example, the cost associated with sending an adjuster to evaluate the damages of a particular claim or the lawyer's fees if the claim is in dispute. IBNR is exactly how it sounds, it accounts for the events that have already occurred (and are attributed to a particular period) but have not been reported to the insurer. A large long term IBNR is common in the case of bodily injury, where accidents happen, but injuries and the resulting claims may not appear for a long time.

To account for these unknown claims, data must be organized properly and a technique that compensates must be followed. As cited in Foundations of Casualty Actuarial Science the three guidelines, which were originally purposed by Berquist and Sherman in a 1977 paper, for the organization of loss data to be used in the estimation of ultimate losses are:

1. Data may be provided by accident year, report year, policy year, underwriting year, or calendar year (in descending order of preference), by development year.

2. The number of years of development should be great enough so that further developments will be negligible.

3. Allocated loss expenses should be included with losses or shown separately; and clearly labeled as such.

Organization of data in this manner facilitates the use of a loss development triangle or a cumulative loss triangle, the latter of which is illustrated in the Exhibit on the next page. The next section, which explains the process of estimation of ultimate losses through the cumulative loss triangle technique, will make several references to this exhibit.

Exhibit. "Estimation of Ultimate Loss with Loss Development Triangles"

Table E1. "Annual by Annual Cumulative Incurred Loss Triangle"

Accident Years	Developed Months						
	12	24	36	48	60	72	84
1982	58,641	74,804	77,323	77,890	80,728	82,280	82,372
1983	63,732	79,512	83,680	85,366	88,152	87,413	
1984	51,779	68,175	69,802	69,694	70,014		
1985	40,143	67,978	75,144	77,947			
1986	55,665	80,296	87,961				
1987	43,401	57,547					
1988	28,800						

Table E2. "Age to Age Ratio Triangle and Averages with Age to Ultimate Factors"

Accident Years	Developed Months					
	12-24	24-36	36-48	48-60	60-72	72-84
1982	1.27563	1.03367	1.00733	1.03644	1.01923	1.00112
1983	1.24760	1.05242	1.02015	1.03264	0.99162	
1984	1.31665	1.02387	0.99845	1.00459		
1985	1.69340	1.10542	1.03730			
1986	1.44249	1.09546				
1987	1.32594					
Avg Last	1.32594	1.09546	1.03730	1.00459	0.99162	1.00112
Age to Ult	1.50260	1.13324	1.03448	0.99728	0.99273	1.00112
Avg Last 2	1.38421	1.10044	1.01788	1.01861	1.00542	1.00112
Age to Ult	1.58967	1.14843	1.04361	1.02528	1.00655	1.00112
Avg Last 3	1.48727	1.07491	1.01863	1.02455	1.00542	1.00112
Age to Ult	1.67939	1.12917	1.05048	1.03126	1.00655	1.00112
Avg Last 4	1.44462	1.06929	1.01581	1.02455	1.00542	1.00112
Age to Ult	1.61819	1.12015	1.04756	1.03126	1.00655	1.00112
Avg Last 5	1.40521	1.06217	1.01581	1.02455	1.00542	1.00112
Age to Ult	1.56356	1.11269	1.04756	1.03126	1.00655	1.00112
Avg M3L5	1.36169	1.06052	1.01512	1.02455	1.00542	1.00112
Age to Ult	1.51176	1.11021	1.04685	1.03126	1.00655	1.00112

Table E3. "Estimated Ultimate Losses as of January 1, 1989 for Each Accident Year"

Method	Accident Year						
	1988	1987	1986	1985	1984	1983	1982
Avg Last	43,275	65,214	90,994	77,735	69,505	87,511	82,372
Avg Last 2	45,782	66,089	91,797	79,918	70,472	87,511	82,372
Avg Last 3	48,366	64,980	92,401	80,384	70,472	87,511	82,372
Avg Last 4	46,604	64,461	92,145	80,384	70,472	87,511	82,372
Avg Last 5	45,031	64,032	92,145	80,384	70,472	87,511	82,372
Avg M3L5	43,539	63,889	92,082	80,384	70,472	87,511	82,372

1.1.1 Cumulative Loss Triangles

Table E1 in the exhibit shows the basic layout of data in a cumulative loss triangle. This layout meets all of the stipulations specified in the guidelines. The data represents *incurred* losses, which are the paid losses (claims that the insurer has actually paid to a recipient) plus an additional case reserve. A case reserve is a dollar amount that an adjuster estimates for the claims reported to the insurer, but the insurer has not actually paid. It is further assumed that all of the data is incurred loss data; thus, creating a homogenous group.

The losses are listed for each accident period by development period, in this case the period is one year. This particular expression of the data is appropriately named an annual by annual (AxA) triangle and satisfies the first of the three guidelines. In order to satisfy the second guideline an assumption is required. In this example, the assumption is made that after seven developed years (84 months) all claims will be reported to the insurer. In reality, the amount of time will vary depending on the product. For instance, auto collision insurance may have all claims reported within two years, but auto bodily injury may take decades before all claims are reported. With this assumption, the cumulative loss triangle meets the second guideline.

The last guideline is for the purpose of clarifying the data. So long as all the data is homogeneous, as already shown, it does not matter whether the losses include allocated loss expenses or not. An estimate of the ultimate loss will result, but there is little use for this estimate if it is unknown exactly what it estimates. On the other hand, if some of the data included allocated expenses while some did not (non-homogeneous data), this estimation technique would lead to erroneous estimates. This again stresses the importance of using the proper data.

The basic layout for this cumulative loss triangle lists the total amount, in dollars (or larger units, such as thousands of dollars), for each accident year at the end of every twelve developed months. In this case, after twelve months, losses total $58,641 for the accidents that occurred in 1982; after twenty four months losses total $74,804; and so on. The entire table is evaluated as of January 1, 1989, which is twelve developed months since the beginning of accident year 1988. Also note that at this time insureds are assumed to have reported to the insurer all claims with events that occurred in accident year 1982.

The next step in this process is formulation of an age to age ratio triangle. An age to age ratio is a number that when multiplied with the loss at a given developed period, results in the loss at the next developed period. To calculate the age to age ratios, start with the second column of the cumulative loss triangle and divide each element by the preceding element of the same accident period. In this example, divide the total losses at twenty four developed months by the total at twelve developed months for each accident year. Then repeat the process for each of the remaining developed months. The age to age ratio triangle of Table E2 shows these results.

1.1.2 Various Averaging Methods

An analysis of the past age to age ratios will result in a series of age to age factors that can "move" current total losses for each accident period forward in time. Moving the losses forward far enough in time will result in an estimate of the ultimate loss. This estimate depends on the method used to formulate the age to age factors, thus some methods may produce better estimates than others. Additionally, this illustrates that enough age to age ratios are required in order for the current dollars to be carried forward far enough to reach their ultimate loss, which emphasizes the need for the second guideline.

A common practice of formulating age to age factors is averaging the past age to age ratios for each developed month. Table E2 presents several different averages and the factors that result. The age to age factors for "Avg Last" are the last age to age ratios for each developed year, while "Avg Last 2" uses the average of the last two ratios as the age to age factor, and so forth. The last average listed, "Avg M3L5," is the average of the middle three ratios of the last five, meaning that the high and low values are excluded. If the number of ratios to be averaged is greater than the number of ratios that are available, the calculation of the factor is reduced to an average of the existing ratios.

Below each row of averages is a row of age to ultimate factors. These values are products of the age to age factor and all proceeding age to age factors. The age to ultimate factors are the values that, when multiplied with the current total losses, will yield an estimate of the ultimate loss. Table E3 lists the estimated ultimate losses for each accident year by the method used to create the age to ultimate factor. For instance, using the method "Avg M3L5," the estimated ultimate loss for accident year 1988 is

$$28,800 \times 1.51176 \approx 43,539$$

and the estimated ultimate loss for accident year 1985 is

$$77,947 \times 1.03126 \approx 80,384.$$

Notice that all of the values of the 1982 column are the same as the total losses, in Table E1, at the last developed month for accident year 1982, which results from the assumption that the insurer will realize all losses after seven years.

The selection of age to age factors is an extremely important aspect in the estimation of ultimate losses. In this example and the following analysis, selection of age to age factors relies completely on the methods of averaging discussed above. This method is adopted for the convenience of calculation and to facilitate comparisons of the different averages. However, in reality, selection of age to age factors may be purely subjective. In the case where ultimate losses are needed for the calculation of reserves, the actuary may select a value for a particular age to age factor that is greater than any of the averages. This may cause the insurer to overstate their required reserves, but on the other hand, it will increase the probability that the company will have enough funds to pay all claims.

1.2 Layout of Data and Beginning Assumptions

A deterministic claims environment model is a useful tool for looking into these estimation methods and finding how assorted factors affect the estimated losses. With a model, it is easy to introduce changes to environmental variables and quickly determine a true ultimate loss. Using the averaging methods discussed in Sub-Section 1.1, the model yields several estimates of the ultimate loss. For the purpose of comparison between averages, the relative error of each estimate is calculated as

$$\text{Relative Error} = \frac{\text{estimate} - \text{ultimate}}{\text{ultimate}}.$$

The relative error of each estimate is the percentage of the true ultimate loss that a given estimate overstates or understates. In addition, the relative error will have a sign indicating the direction of the error, with a positive sign representing an overstatement and a negative showing an understatement.

In order to evaluate between different hypothetical environments, the mean error is computed as the arithmetic mean of the relative errors over all of the averaging methods. However, this does not necessarily paint an accurate picture of the effects different environments have on the estimation techniques. To further analyze these effects, the sample variance of the errors is found with the formula

$$\text{Error Variance} = \frac{\sum_{i=1}^{n} (\text{relative error}_i - \text{mean error})^2}{n-1},$$

where n is equal to the number of averaging methods. The sample variance is a measure of the dispersion of the estimates. A small variance indicates that there might be no significant difference between the averaging methods while a larger variance could show that some methods are better than others for a particular environment.

The basis of the model is a series of ultimate paid loss dollars for each of fifteen years' accident quarters. Again, paid loss is the total amount that the insurer has actually paid to settle claims at any given time. This simplifies the analysis by removing the need to calculate a case reserve as described at the beginning of Sub-Section 1.1. Assuming that the ultimate loss is the product of three environmental variables enables changes in the ultimate loss from quarter to quarter to be expressed as changes in these individual variables.

The product of the first two variables, earned exposure and frequency, yields the total number of claims for each accident quarter. Earned exposure is the number exposed to risk, for example the number of autos an insurer covers. Frequency is the percentage of units exposed to risk resulting in a claim. The result of this product is multiplied by the average severity, or average dollar amount, of a claim to produce the ultimate loss for a particular accident quarter. The calculations are illustrated as follows:

$$\text{claims}_i = \text{earned exposure}_i \times \text{frequency}_i$$

and

$$\text{ultimate loss}_i = \text{claims}_i \times \text{severity}_i,$$

where i is the accident quarter.

A series of annual growth rates and an initial value represents the change from quarter to quarter of the environmental variables. Another way to express this uses the idea of compound interest, the variables compound quarterly based on an annual rate. Therefore the variable for each successive quarter is

$$\text{variable}_i = \text{variable}_{i-1} \times \left\{1 + 4\left[(1 + \text{rate}_j)^{\frac{1}{4}} - 1\right]\right\},$$

where i is the current accident quarter and j is the current accident year.

The next variable portion of the model is a 60 by 60 pattern matrix. Initially held constant, this matrix contains the percentage of the ultimate loss developed in each accident quarter. The AxA triangle consists of sums of all currently developed dollars, an example of which is illustrated in Figure 1.1. The values enclosed within the bold outline are those that are a part of the sum for the first developed year of the first accident year. The diagonal nature of the region results from the same principles of the loss development triangles. At the end of accident quarter 4, the claims associated with accident quarter 4

Figure 1.1 "Example of converting Quarter by Quarter Cummulative Paid Loss Data into Annual by Annual Cummulative Paid Loss Data"

Accident	***Developed Quarters***					
Quarters	***@1***	***@2***	***@3***	***@4***	***@5***	***@6***
1	5	10	15	$\boxed{20}$	25	. .
2	4	7	$\boxed{14}$	20	26	. .
3	6	$\boxed{11}$	16	21	24	. .
4	$\boxed{7}$	10	15	19	25	. .
5	5	9	12	17	23	. .
6	:	:	:	:	:	

have been paid by the insurer, plus claims associated with the previous three quarters. Therefore the region is the updated total dollars paid associated with that given accident year. As discussed in the following section, different types of policies and social trends can be illustrated by varying the values of this matrix. These "moving" models encompass a much larger scope of variables than the constant models.

1.3 Short-tail and Long-tail Patterns

An environment where insureds report claims quickly and the insurer likewise pays the claims quickly is appropriately named a short-tail payment pattern. An example of a coverage that reflects this pattern is personal auto collision. Considering the nature of an auto accident, where a car is either repaired or replaced promptly (usually not fast enough for the insured), insureds will report the bulk of claims in the first few developed quarters and the remainder of claims will tail off quickly. The tail, or delayed payment of the claim, is likely a result of disputed fault. If the claim is not in dispute, the insurer is able to pay swiftly since the damages resulting from an auto accident are specific and the loss is relatively simple to estimate.

In a coverage such as bodily injury, claims are reported quickly, but usually are not settled for an extended period of time. Settlement delays are often the result of the slow process of contested claims or having to wait until all damages have been assessed (length of time in the hospital, amount of rehabilitation for the injured party, etc.). Since a long delay until payment exists (unlike collision coverage) the insurer will set aside a case reserve, which is the adjuster's estimate as to the future amount the insurer will have to pay on the claim. However, it is not the intent of this paper to analyze the effect these case reserves have on the estimates of ultimate loss. Therefore the long-tail payment pattern reflects the time that the loss is actually paid. Additionally for the purpose of this model, all claims are assumed to have been paid by the end of the fourteenth developed year to satisfy the second guideline of Sub-Section 1.1.

In reality, these patterns change over time as a result of several factors. For instance, change in public attitude may cause people to sue for more damages. The litigation process will then lengthen the time until settlement. However, such a change is gradual and is reflected in the payment pattern over several years as an increase in the length of time until an accident period reaches ultimate. Furthermore, the manner in which an insurance company handles claim processing and adjustment can shorten or lengthen the time until the claim is paid. A change by the company similar to this shows up in the payment pattern suddenly and will increase the error in estimating the ultimate loss. The actuary has the responsibility to investigate instances such as these and take them into account when selecting age to age factors.

1.4 Summary of Adopted Notation

All hypothetical environments are given names representing the growth rates for each environmental variable. This name consists of three characters listed in brackets; the first is the growth of the earned exposure, the second is the growth of the frequency of claims, and the third is the growth of the average severity. The four different patterns of growth discussed in this paper are as follows:

- z = zero growth
- c = constant growth
- a = accelerating growth
- d = decelerating growth.

For example, the name [z, z, z] is the "no growth" model where all accident quarters behave exactly the same. Further, the name [d, z, c] represents an environment with decelerating exposure growth, zero frequency growth, and constant severity growth. Table 1.1 lists the actual percentages used in the models.

Table 1.1. "The Growth Parameters and their Annual Values"

Year	***z***	***c***	***a***	***d***
1	0.00%	0.05%	0.01%	0.15%
2	0.00%	0.05%	0.02%	0.14%
3	0.00%	0.05%	0.03%	0.13%
4	0.00%	0.05%	0.04%	0.12%
5	0.00%	0.05%	0.05%	0.11%
6	0.00%	0.05%	0.06%	0.10%
7	0.00%	0.05%	0.07%	0.09%
8	0.00%	0.05%	0.08%	0.08%
9	0.00%	0.05%	0.09%	0.07%
10	0.00%	0.05%	0.10%	0.06%
11	0.00%	0.05%	0.11%	0.05%
12	0.00%	0.05%	0.12%	0.04%
13	0.00%	0.05%	0.13%	0.03%
14	0.00%	0.05%	0.14%	0.02%
15	0.00%	0.05%	0.15%	0.01%

2 *The Deterministic Claims Model*

Before creating a claims environment that "moves" in time, one must begin with a static model, which has zero growth between years and constant report and payment patterns. For the sake of simplicity, a quarterly earned exposure of 1,000,000 with a quarterly claim frequency of 0.001 were chosen as initial values to begin testing the model. With zero growth for all environmental variables, 1,000 claims is the expected ultimate count. With a uniform payment pattern (1/56 claims per quarter for 56 quarters and zero for the remaining four, which allows an additional year to satisfy the second guideline of Sub-Section 1.1), the paid loss matrix results in 17.86 paid claims developed per quarter.

This very basic model makes it possible to check calculations in all of the spreadsheet formulas and pick out instances where errors occur. With zero growth, all of the age to age ratios should be equivalent and thus each averaging method results in equivalent estimates, which are equal to the true ultimate. The next step in analyzing the consistency of the basic model is to vary the growth factors and survey the results of both loss development triangles. Since the payment pattern remains uniform, the estimates should fall relatively close to the true ultimate loss.

It is interesting to note the error that results from the finite mantissa of the floating point form that the spreadsheet uses to hold real numbers. For a detailed description of the mantissa and related errors, consult the Numerical Analysis text listed in the Bibliography at the end of this paper. Theoretically, if an environmental variable is increasing at a positive constant rate, as in [c, z, z], [z, c, z] and [z, z, c], the calculation of the growth of any single environmental variable is

$$\text{variable}_i = \text{variable}_{i-1} \times \{1 + [(1 + 0.5)^{\frac{1}{4}} - 1]\}.$$

The calculation of the ultimate loss in the i^{th} accident quarter when two variables have zero growth and one has constant growth is

$$\text{ultimate loss}_i = \text{earned exposure}_{i-1} \times \text{frequency}_{i-1} \times \text{severity}_{i-1} \times [1 + 4 \times (1.05^{\frac{1}{4}} - 1)].$$

This second equation illustrates that the commutative property of multiplication allows the growth on one element to be expressed as the growth on another. Therefore, one would assume that the three scenarios mentioned above would all produce equal estimates and equal errors. However, Table 2.1 shows that the mean error of [c, z, z] is negative, while that of both [z, c, z] and [z, z, c] are positive. Although the values appear as zero in the table when rounded to three decimal places, a negative sign indicates a value that is less than zero. This difference in the errors result from multiplying the large number of exposures by the growth rates, which prevents the spreadsheet from carrying as many decimal places of accuracy. Since some insurers realize ultimate losses of billions of dollars, even this small error could result in estimates that are off by thousands of dollars.

Another initial observation that one can make at this time, is the effect of a constant change versus an accelerating change. As will be discussed later in the analysis, an accelerating change in an environmental variable may produce much greater errors than a constant change, as shown in Table 2.1.

However, it does not appear to be as important to look at how individual variables are changing, but at how the ultimate losses are changing. The various environmental growth patterns are able to counter act each other's effects, and possibly slow the rate of

Table 2.1 "Sample Results of Changing Environment Variables on AxA Paid Loss with a Uniform Payment Pattern"

Growth Factors			Final Year	AxA Triangle Estimated Ultimate Losses Above Relative Error						Mean	Error
Exp	Freq	Sev	Ultimates	Avg Last Relative Error	Avg Last 2 Relative Error	Avg Last 3 Relative Error	Avg Last 4 Relative Error	Avg Last 5 Relative Error	Avg M3L5 Relative Error	Error	Variance
z	z	z	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
c	z	z	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854		
				-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	0.00000
z	c	z	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
z	z	c	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854		
				-0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
z	z	a	5,952,998	5,935,804	5,932,608	5,929,411	5,926,214	5,923,021	5,923,053		
				-0.289%	-0.343%	-0.396%	-0.450%	-0.504%	-0.503%	-0.414%	0.00000
c	z	a	12,159,084	12,123,880	12,117,316	12,110,748	12,104,181	12,097,620	12,097,685		
				-0.290%	-0.344%	-0.398%	-0.452%	-0.505%	-0.505%	-0.415%	0.00000

increase or decrease of the ultimate losses. For instance, a development such as the driver's side air bag in a rapidly growing economy, might help produce better estimates. This could result because the air bag has been shown to reduce the average severity of a bodily injury claim, which may be sufficient to counter act the effect of the economy's growth.

2.1 Constant Short-tail Pattern

The development of a model with a constant short-tail pattern simply entails substituting a new pattern matrix for the previous one, where the rows must sum to one or one hundred percent of the future claims. The matrix consists of the values

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
72.0%	18.0%	1.2%	4.8%	1.0%	1.0%	0.5%	0.5%	0.5%	0.5%

where the number in the shaded region represents the quarter in which the percentage was developed. In other words, in the short-tail pattern, all claims are realized and paid within two and a half years.

With the creation of the short-tail model, several more patterns of growth parameters are introduced and allow for further analysis of the possible effects. Table 2.2 lists several possibilities beginning with [c, c, c] and ending with [a, z, d]. The purpose of this series is to illustrate that the manner in which individual variables change is not as important as how ultimate losses change.

In comparing the relative errors, one should notice that in Table 2.2 the large errors stem from environments where variables are accelerating or decelerating, such as [c, c, c] compared with [c, z, a]. Even with all variables increasing at a constant rate, the

Table 2.2 "Results of Changing Environment Variables on AxA Paid Loss with a Short-tail Payment Pattern"

Growth Factors			Final Year	AxA Triangle Estimated Ultimate Losses Above Relative Error						Mean	Error
				Avg Last	Avg Last 2	Avg Last 3	Avg Last 4	Avg Last 5	Avg M3L5		
Exp	Freq	Sev	Ultimates	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Error	Variance
z	z	z	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
c	z	z	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
c	c	c	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
a	z	z	5,952,998	5,951,481	5,950,784	5,950,084	5,949,379	5,948,671	5,948,679		
				-0.025%	-0.037%	-0.049%	-0.061%	-0.073%	-0.073%	-0.053%	0.00000
z	z	d	6,245,213	6,246,983	6,247,784	6,248,579	6,249,371	6,250,158	6,250,166		
				0.028%	0.041%	0.054%	0.067%	0.079%	0.079%	0.058%	0.00000
a	z	d	18,590,891	18,591,489	18,591,719	18,591,920	18,592,093	18,592,238	18,592,295		
				0.003%	0.004%	0.006%	0.006%	0.007%	0.008%	0.006%	0.00000
c	z	a	12,159,084	12,155,972	12,154,541	12,153,102	12,151,657	12,150,203	12,150,218		
				-0.026%	-0.037%	-0.049%	-0.061%	-0.073%	-0.073%	-0.053%	0.00000

effect of acceleration in the latter is evident. This results from a limitation of the estimation techniques, or actually with the averages themselves. Looking at the graph of Figure 2.1 shows two different environments that reach the same ultimate loss. When the age to age ratios of the last few developed years are averaged, the estimates of the true ultimate loss will lie along a secant line. Therefore the shape of the curve directly affects whether and by how much the estimates will be overstated or understated.

Figure 2.2 illustrates two different ultimate loss curves with sample secant lines, one where the ultimate losses are accelerating and one where they are decelerating. This figure clearly shows that in an accelerating environment, the estimate for the ultimate loss of the last accident year will be understated. If the curve is concave up, then no matter where the secant line lies, the point with the estimate of ultimate loss for the final accident year will always be below the point on the curve. The opposite is true for the concave down curve of a decelerating environment. This conclusion is confirmed in Table 2.2, where the mean relative error is negative for [a, z, z] and positive for [z, z, d].

This leads to the concept of how growth factors can counter act each other. In Table 2.2, the situations with just one variable accelerating or decelerating with all others at zero growth ([a, z, z] and [z, z, d]) have a rather large absolute relative error when compared to a combination of one variable accelerating and one decelerating with the other at zero growth ([a, z, d]). Although this last hypothetical environment contains more variability, since one factor is accelerating while the other is decelerating. Over time the effect of the individual variables counter act each other and result in better estimates.

Figure 2.1 "Sample Ultimate Loss Curves"

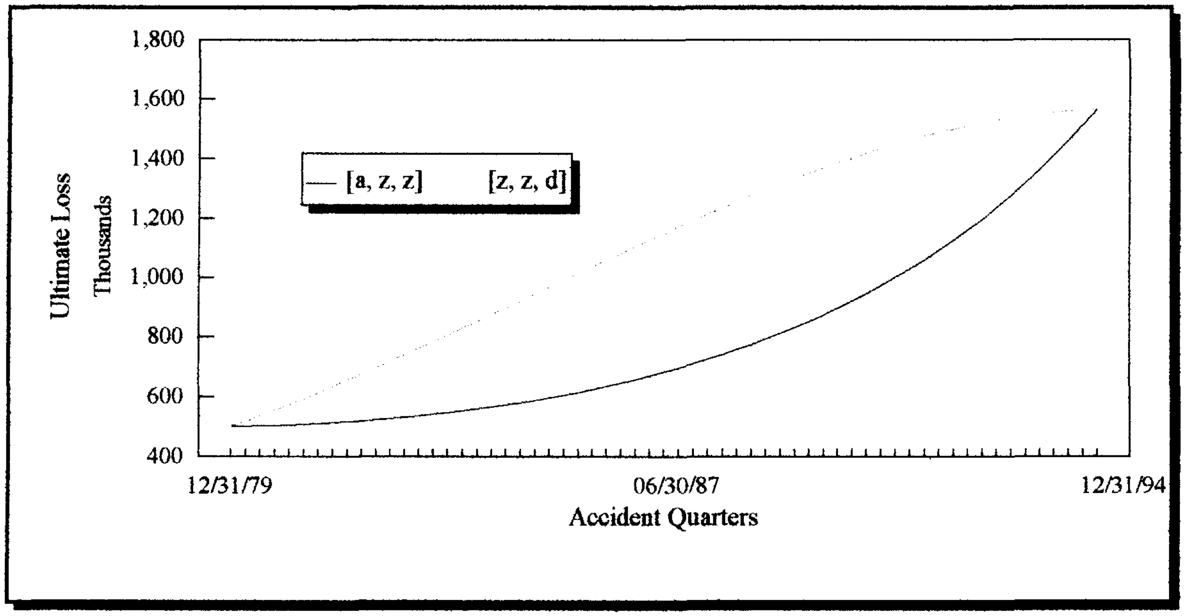
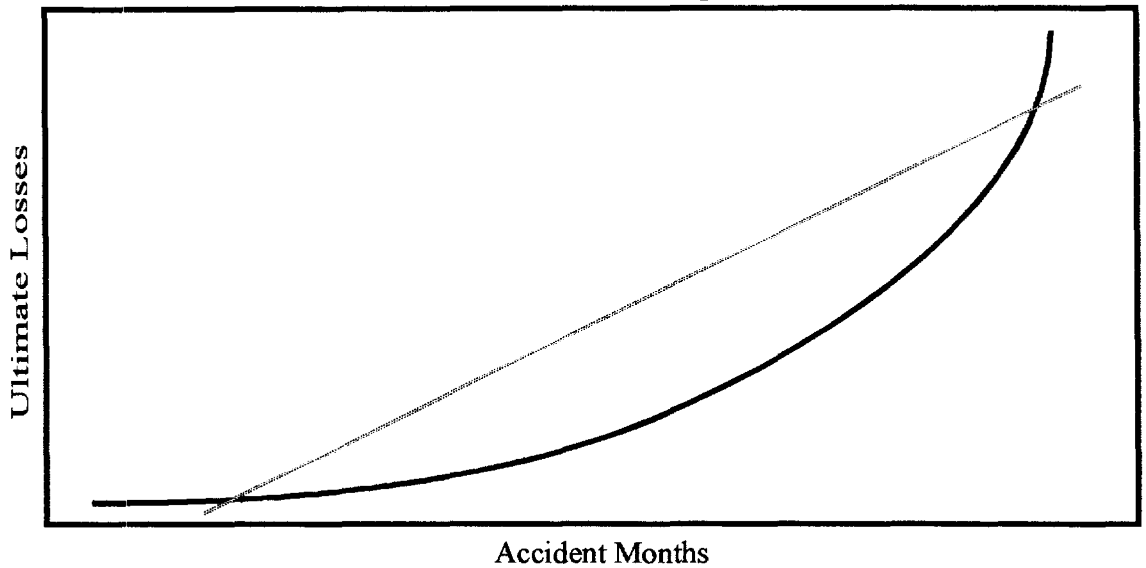
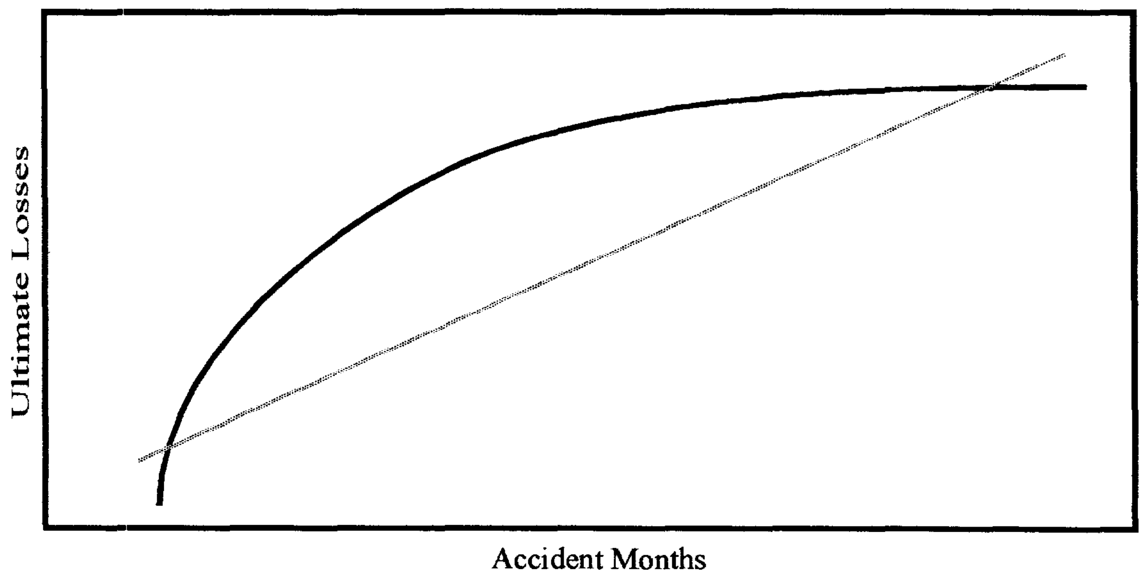


Figure 2.2 "Behaviors of Ultimate Losses"

Accelerating



Decelerating



2.2 Constant Long-tail Pattern

The development of the constant long-tail pattern is exactly the same as the short-tail and uniform patterns. In this case the percentage of the ultimate paid loss for each developed quarter is

1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
16.5%	24.0%	13.5%	16.5%	7.5%	5.1%	3.3%	2.7%	2.4%	1.8%
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
1.2%	0.9%	0.6%	0.6%	0.6%	0.45%	0.3%	0.3%	0.3%	0.3%
21	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
0.09%	0.15%	0.09%	0.09%	0.06%	0.06%	0.06%	0.06%	0.00%	0.06%
31	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
0.06%	0.06%	0.06%	0.00%	0.00%	0.00%	0.00%	0.06%	0.00%	0.06%
41	**42**	**43**	**44**	**45**					
0.00%	0.06%	0.00%	0.06%	0.01%					

where again, the number in the shaded region represents the quarter in which the percentage below it is realized and paid by the insurer. With this pattern, a little over eleven years must develop before an accident quarter reaches its ultimate loss.

After testing the same combinations of growth factors as those listed in Table 2.2 for the short-tail pattern, an interesting observation results. Estimation in an environment utilizing a short-tail pattern engenders estimates that are closer to the true ultimate loss than when a long-tail payment pattern exists. A comparison of each of the hypothetical environments of Table 2.2 with those of Table 2.3 bears this out. The relative errors of the short-tail environments seem significantly less than those of the long-tail environments. This again shows a limitation of the estimation technique.

As an example, if paid losses are experiencing an increasing trend (the curve is concave up), averaging the last five or middle three of the last five age to age ratios will

Table 2.3 "Results of Changing Environment Variables on AxA Paid Loss with a Long-tail Payment Pattern"

Growth Factors			Final Year	AxA Triangle Estimated Ultimate Losses Above Relative Error						Mean	Error
Exp	Freq	Sev	Ultimates	Avg Last Relative Error	Avg Last 2 Relative Error	Avg Last 3 Relative Error	Avg Last 4 Relative Error	Avg Last 5 Relative Error	Avg M3L5 Relative Error	Error	Variance
z	z	z	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000	2,000,000		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
c	z	z	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854	4,082,854		
				-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	-0.000%	0.00000
c	c	c	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481	17,024,481		
				0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.000%	0.00000
a	z	z	5,952,998	5,945,458	5,942,233	5,938,994	5,935,741	5,932,476	5,932,506		
				-0.127%	-0.181%	-0.235%	-0.290%	-0.345%	-0.344%	-0.254%	0.00000
z	z	d	6,245,213	6,253,979	6,257,675	6,261,354	6,265,015	6,268,655	6,268,691		
				0.140%	0.200%	0.258%	0.317%	0.375%	0.376%	0.278%	0.00000
a	z	d	18,590,891	18,593,786	18,594,836	18,595,754	18,596,541	18,597,197	18,597,458		
				0.016%	0.021%	0.026%	0.030%	0.034%	0.035%	0.027%	0.00000
c	z	a	12,159,084	12,143,598	12,136,970	12,130,313	12,123,626	12,116,915	12,116,977		
				-0.127%	-0.182%	-0.237%	-0.292%	-0.347%	-0.346%	-0.255%	0.00000

result in an understated age to age factor. Since the estimate of the ultimate loss is based on the product of several of these age to age factors, a greater number of terms in the product will lead to a greater understatement. For further verification, look back to the [c, z, a] environment of Table 2.1. The uniform payment pattern spans fourteen years with about twice the error of the long-tail pattern which covers approximately eleven years.

2.3 The "Moving" Model

The last topic of this analysis is the idea of the "moving" model. In these models both the payment patterns and the growth factors are varied. Therefore, all aspects of the loss development triangles move in time and contribute to a better representation of errors that may occur in an actual environment. For the sake of illustration, the movement of the payment pattern will simply be a linear lengthening or shortening of the patterns used in the previous models.

The development of the increasing models begins with converting the incremental payment patterns (both short-tail and long-tail) into cumulative patterns. Then a new cumulative matrix is created containing values that uniformly approach 1 over each accident quarter. The key to this matrix is that the number of developed quarters until a particular accident quarter reaches 1 is uniformly increased up to the new length of time. For example, in the case of a long-tail paid loss, if it is desired to increase the pattern from eleven years to fourteen years, the first row will reach 1 immediately (no change) while the last row will reach 1 after fourteen years (56 quarters) with a uniform progression between rows.

This new matrix represents the shifting of dollars that would previously been paid early in the payment pattern to later points in the pattern. The new pattern is simply the product of each cell of the cumulative payment pattern and the corresponding cell of the matrix described above, then converted back to an incremental pattern. In the case of the short-tail pattern, the procedure is exactly the same, except that the pattern is increased an additional two years from ten quarters to eighteen quarters.

To speed up calculations, the decreasing models are formed by vertically flipping the increasing models' pattern matrices. To accomplish this, a vertically flipped identity matrix (all cells are zero except for a diagonal of 1's starting at the lower left corner of the matrix) is multiplied by the incremental payment pattern matrix. After forming these four new patterns, the calculation of estimates proceeds in the same manner as the previous models.

Tables 2.4 through 2.7 show the results of changing the payment pattern on the hypothetical environments previously discussed. Generally, if a pattern is increasing over time, the estimates will be understated. This is a result of the fact that the estimate is formed by multiplying some current dollar amount by an age to ultimate factor. As the pattern lengthens, this current dollar amount decreases and the age to ultimate factor, which is based on prior years of experience, is understated relative to the current dollars. Multiplying this understated factor by the current dollars produces an understated estimate. The reverse is true for the decreasing patterns.

Table 2.4 "Results of Changing Environment Variables on AxA Paid Loss with a Lengthening Short-tail Payment Pattern"

Growth Factors Exp	Growth Factors Freq	Growth Factors Sev	Final Year Ultimates	AxA Triangle Estimated Ultimate Losses Above Relative Error: Avg Last Relative Error	Avg Last 2 Relative Error	Avg Last 3 Relative Error	Avg Last 4 Relative Error	Avg Last 5 Relative Error	Avg M3L5 Relative Error	Mean Error	Error Variance
z	z	z	2,000,000	1,908,196	1,896,658	1,885,430	1,872,984	1,861,256	1,861,304		
				-4.590%	-5.167%	-5.728%	-6.351%	-6.937%	-6.935%	-5.951%	0.00009
c	z	z	4,082,854	3,895,160	3,871,707	3,848,740	3,823,307	3,799,335	3,799,472		
				-4.597%	-5.172%	-5.734%	-6.357%	-6.944%	-6.941%	-5.957%	0.00009
c	c	c	17,024,481	16,239,469	16,142,511	16,046,378	15,940,124	15,839,918	15,840,825		
				-4.611%	-5.181%	-5.745%	-6.369%	-6.958%	-6.953%	-5.970%	0.00009
a	z	z	5,952,998	5,676,483	5,641,696	5,607,389	5,569,544	5,533,844	5,534,082		
				-4.645%	-5.229%	-5.806%	-6.441%	-7.041%	-7.037%	-6.033%	0.00010
z	z	d	6,245,213	5,960,880	5,925,870	5,891,585	5,853,489	5,817,594	5,817,852		
				-4.553%	-5.113%	-5.662%	-6.272%	-6.847%	-6.843%	-5.882%	0.00009
a	z	d	18,590,891	17,734,343	17,628,764	17,523,960	17,408,081	17,298,769	17,299,849		
				-4.607%	-5.175%	-5.739%	-6.362%	-6.950%	-6.944%	-5.963%	0.00009
c	z	a	12,159,084	11,593,432	11,522,675	11,452,465	11,375,087	11,302,074	11,302,681		
				-4.652%	-5.234%	-5.811%	-6.448%	-7.048%	-7.043%	-6.040%	0.00010

Table 2.5 "Results of Changing Environment Variables on AxA Paid Loss with a Shortening Short-tail Payment Pattern"

Growth Factors Exp	Growth Factors Freq	Growth Factors Sev	Final Year Ultimates	AxA Triangle Estimated Ultimate Losses Above Relative Error: Avg Last Relative Error	Avg Last 2 Relative Error	Avg Last 3 Relative Error	Avg Last 4 Relative Error	Avg Last 5 Relative Error	Avg M3L5 Relative Error	Mean Error	Error Variance
z	z	z	2,000,000	2,015,070	2,026,361	2,042,017	2,058,281	2,075,208	2,072,755		
				0.753%	1.318%	2.101%	2.914%	3.760%	3.638%	2.414%	0.00015
c	z	z	4,082,854	4,113,899	4,136,995	4,169,082	4,202,352	4,236,913	4,231,980		
				0.760%	1.326%	2.112%	2.927%	3.773%	3.652%	2.425%	0.00015
c	c	c	17,024,481	17,156,262	17,252,955	17,387,811	17,527,088	17,671,256	17,651,308		
				0.774%	1.342%	2.134%	2.952%	3.799%	3.682%	2.447%	0.00015
a	z	z	5,952,998	5,997,422	6,030,444	6,076,668	6,124,452	6,174,002	6,167,030		
				0.746%	1.301%	2.077%	2.880%	3.712%	3.595%	2.385%	0.00015
z	z	d	6,245,213	6,294,228	6,330,396	6,380,354	6,432,183	6,485,995	6,478,394		
				0.785%	1.364%	2.164%	2.994%	3.855%	3.734%	2.483%	0.00016
a	z	d	18,590,891	18,735,516	18,841,380	18,988,948	19,141,286	19,298,897	19,277,213		
				0.778%	1.347%	2.141%	2.961%	3.808%	3.692%	2.455%	0.00016
c	z	a	12,159,084	12,250,638	12,318,217	12,413,002	12,510,789	12,612,010	12,597,991		
				0.753%	1.309%	2.088%	2.893%	3.725%	3.610%	2.396%	0.00015

Table 2.6 "Results of Changing Environment Variables on AxA Paid Loss with a Lengthening Long-tail Payment Pattern"

Growth Factors			Final Year	AxA Triangle Estimated Ultimate Losses Above Relative Error						Mean	Error
				Avg Last	Avg Last 2	Avg Last 3	Avg Last 4	Avg Last 5	Avg M3L5		
Exp	Freq	Sev	Ultimates	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Error	Variance
z	z	z	2,000,000	1,800,422	1,792,527	1,784,706	1,777,505	1,770,492	1,769,908		
				-9.979%	-10.374%	-10.765%	-11.125%	-11.475%	-11.505%	-10.870%	0.00004
c	z	z	4,082,854	3,675,266	3,659,068	3,643,129	3,628,417	3,614,112	3,612,887		
				-9.983%	-10.380%	-10.770%	-11.130%	-11.481%	-11.511%	-10.876%	0.00004
c	c	c	17,024,481	15,323,567	15,255,351	15,189,121	15,127,671	15,068,117	15,062,735		
				-9.991%	-10.392%	-10.781%	-11.142%	-11.491%	-11.523%	-10.887%	0.00004
a	z	z	5,952,998	5,350,941	5,324,400	5,298,327	5,274,027	5,250,343	5,248,567		
				-10.114%	-10.559%	-10.997%	-11.406%	-11.803%	-11.833%	-11.119%	0.00005
z	z	d	6,245,213	5,630,452	5,608,875	5,587,761	5,568,438	5,549,727	5,547,843		
				-9.844%	-10.189%	-10.527%	-10.837%	-11.136%	-11.166%	-10.617%	0.00003
a	z	d	18,590,891	16,736,143	16,662,518	16,591,012	16,524,576	16,460,107	16,454,438		
				-9.977%	-10.373%	-10.757%	-11.115%	-11.461%	-11.492%	-10.862%	0.00004
c	z	a	12,159,084	10,928,807	10,874,320	10,821,110	10,771,402	10,723,023	10,719,298		
				-10.118%	-10.566%	-11.004%	-11.413%	-11.811%	-11.841%	-11.125%	0.00005

Table 2.7 "Results of Changing Environment Variables on AxA Paid Loss with a Shortening Long-tail Payment Pattern"

Growth Factors			Final Year	AxA Triangle Estimated Ultimate Losses Above Relative Error						Mean	Error
				Avg Last	Avg Last 2	Avg Last 3	Avg Last 4	Avg Last 5	Avg M3L5		
Exp	Freq	Sev	Ultimates	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Relative Error	Error	Variance
z	z	z	2,000,000	2,192,969	2,304,561	2,339,799	2,355,997	2,365,376	2,397,141		
				9.648%	15.228%	16.990%	17.800%	18.269%	19.857%	16.299%	0.00129
c	z	z	4,082,854	4,476,781	4,704,610	4,776,553	4,809,622	4,828,769	4,893,622		
				9.648%	15.228%	16.991%	17.800%	18.269%	19.858%	16.299%	0.00129
c	c	c	17,024,481	18,667,017	19,617,184	19,917,224	20,055,137	20,134,974	20,405,460		
				9.648%	15.229%	16.992%	17.802%	18.271%	19.860%	16.300%	0.00129
a	z	z	5,952,998	6,519,082	6,847,154	6,948,074	6,992,343	7,016,314	7,111,042		
				9.509%	15.020%	16.716%	17.459%	17.862%	19.453%	16.003%	0.00122
z	z	d	6,245,213	6,857,388	7,210,640	7,325,218	7,380,250	7,413,941	7,514,020		
				9.802%	15.459%	17.293%	18.175%	18.714%	20.316%	16.627%	0.00137
a	z	d	18,590,891	20,387,728	21,426,703	21,755,499	21,907,068	21,995,054	22,291,196		
				9.665%	15.254%	17.022%	17.838%	18.311%	19.904%	16.332%	0.00130
c	z	a	12,159,084	13,315,211	13,985,314	14,191,414	14,281,791	14,330,700	14,524,211		
				9.508%	15.019%	16.714%	17.458%	17.860%	19.452%	16.002%	0.00122

Looking back at Tables 2.1 through 2.3, the error variances for the hypothetical environments of each model are all zero when rounded to five decimal places. This means that the error variances are less than 0.000005 if not truly zero (as is the case for the "no growth" environment, [z, z, z]). This suggests that there is little difference between the methods and that all of them are likely to produce good estimates.

When surveying the results of Tables 2.4 through 2.7, one will notice that the error variances for the "moving" models are significantly higher than those of the other models. Although these variances are not extremely large, one should be aware of the increase caused by the changing patterns. In particular, the variances of the shortening patterns are higher than those of the lengthening pattern. With these higher variances, different methods could result in significantly different estimates.

For example, look at the [z, z, d] environment of Table 2.7. In this case, the error variance of the estimates produced by the various averaging methods is 0.00137, which corresponds to a standard deviation of 0.03701. This means that approximately 68%, which is just over two thirds, of the estimates fall within plus or minus 3.701% of the mean error. With an ultimate loss of $6,245,213, the distance between the extremes of this range is approximately $462,315. So taking into account only about two thirds of the estimates there is already a large dispersion of the estimated ultimate losses.

3 *A Final Word*

This analysis only brushes the surface of estimation techniques in the property and casualty insurance industry. However, these models are easily adaptable to different strategies and facilitate a study similar to that of Section 2. This section contains a discussion of a practical claims environment and will explain some ideas on the development of a future generation of models.

Besides testing several estimation strategies, a subsequent goal of this paper is to illustrate the usefulness of models and computer simulations. Chapter Four of Foundations of Casualty Actuarial Science sets forth the following four phase strategy for the estimation of the ultimate loss:

1. Review of the data to identify its characteristics and possible anomalies. Balancing of data to other verified sources should be undertaken at this point.

2. Application of appropriate reserve estimation techniques.

3. Evaluation of the conflicting results of the various methods used, with an attempt to reconcile or explain the bases for different projections. At this point the proposed reserving ultimates are evaluated in contexts outside of their original frame of analysis.

4. Prepare projections of reserve development that can be monitored over the subsequent calendar periods. Deviations of actual from projected developments of counts or amounts is one of the most useful diagnostic tools in evaluating accuracy of reserve estimates.

A well designed model could become an important tool for each of these four steps. The types of models analyzed through out this paper are especially applicable to the first and third steps. In the first step, they help characterize types of data by showing how combinations of policy type and environmental changes affect the ultimate losses. The error analysis helps to "explain the bases for different projections" as required by the third step.

3.1 Conclusion

The hypothetical environments of Tables 2.1 through 2.7 are primarily designed to clearly demonstrate the ideas stated in each corresponding sub-section. But in conclusion, this paper will profit from a practical application of the models. To begin, one must attempt to fit a series of parameters to the growth factors that replicate trends over the past fifteen years for a particular line of insurance. For example, in the line of personal auto collision insurance, the earned exposure is the number of autos insured by an insurer over a given accident quarter. As an economy expands, more and more people purchase autos, and due to state regulations, must purchase insurance. Further, the growth of suburbs around many cities and the desire to commute increases the need for autos. Therefore a constant annual increase that is slightly higher than the gross national product is a logical assumption. The gross national product averages about a three percent annual growth, so a five percent annual growth of the earned exposure seems to fit prior experience.

The technology involved with the manufacturing and maintenance of these new autos is increasing as rapidly as the technology of computers and other household items is in today's society. With this increasing technology come increasing costs, so the cost of replacement or repair of a damaged auto is also changing at a rapid pace. Consequently, one will assume that the average severity will accelerate over time. However, the frequency will be held at zero. Since on an annual basis, the frequency seems to remain about the same over time. If the insurer plans to use quarter by quarter loss development, the frequency may vary within a year. For instance, winter and summer months may realize a higher frequency of accidents with hazardous roads or frequent travel.

With all of the parameters established, a model can then be created to test the estimates. In the notation of this paper, this proposed environment is [c, z, a] and appears on the last line of Tables 2.1 through 2.7. As stated earlier, if collision insurance claims are reported and paid quickly, the short-tail pattern of Table 2.2 yields the most applicable results. In a similar environment, the property and casualty actuary can expect a mean understatement of 0.053% and will be able to adjust the methods accordingly.

3.2 Suggestions for Further Study

The models created for this project deal primarily with constants. For example, constant patterns, constant growth rates, and even in the case of accelerating growth rates, the acceleration is at a constant rate. Introducing several elements to create variability from quarter to quarter in the growth factors and patterns will provide a better representation of the accuracy of loss development triangles in the estimation of ultimate loss. Perhaps this can be accomplished by taking the deterministic model a step further.

The next stage of this project could be the development of a stochastic model to simulate a claims environment. With a simulation, one can produce enough data to estimate the ultimate loss and then carry the model far enough forward to reach the true ultimate loss. The methods of stochastic processes described in <u>Actuarial Mathematics</u>

seem applicable to this situation. However, this would be a complicated process and a couple issues need to be discussed.

The first item that should be addressed is that an insurer pays larger claims later, while the smaller claims are settled earlier. This is a result of the nature of large claims, in that they tend to be more complicated than smaller claims and that they usually require some form of litigation. Since these large dollar amounts do not settle until later in the tail, they may not have developed as of the current accident period. This will increase errors when using a loss development triangle to estimate the current ultimate loss.

Models of this type should take into account many separate variables. A fully stochastic model would have to involve several random variables, which significantly complicates development and testing. A delicate balance exists between the inaccuracies of a deterministic model such as those of this analysis and an overly complicated stochastic model. The actuary must carefully judge the purpose of the model to decide if the added time spent developing a stochastic model would be appropriate.

Acknowledgments

This paper and the models discussed herein were written in completion of the Honors College requirements at Ball State University during the Spring semester of 1995, which completes my undergraduate degree in Actuarial Science. Dr. Rebecca L. Pierce, an outstanding member of BSU's Mathematical Sciences faculty, served as my thesis advisor, providing useful suggestions and lots of moral support. I would also like to thank the rest of the mathematics department for the valuable education they provided, in particular the Actuarial Science Professors Dr. John A. Beekman and Mr. W. Bart Frye. I must also give my regards to Nationwide Insurance for the internship where I was able to study the aspects of the property and casualty insurance industry and lay the ground work for this thesis. I am further indebted (literally) to my parents, Paul and Alice Pleva, for their support and the financial assistance they provided for my undergraduate degree.

Bibliography

Bowers, N.L., Gerber, H.U., Hickman, J.C., Jones, D.A., and Nesbitt, C.J. Actuarial Mathematics. Society of Actuaries: 1986. pp. 317-395.

Burden, Richard L., and Faires, J. Douglas. Numerical Analysis. PWS-Kent Publishing Company, Boston: 1993. Fifth Edition. pp. 12-21.

Fisher, W.H., and Lester, E.P. "Loss Reserve Testing in a Changing Environment." Proceedings of the Casualty Actuarial Society. Casualty Actuarial Society: 1975. Vol. LXII, pp. 154-171.

Hogg, Robert V. and Tanis, Elliot A. Probability and Statistical Inference. Macmillan Publishing Company, New York: 1988. Third Edition. pp. 23-25.

Standard, James N. "A Simulation Test of Prediction Errors of Loss Reserve Estimation Techniques." Proceedings of the Casualty Actuarial Society. Casualty Actuarial Society: 1985. Vol. LXXII, pp. 149-153.

Wiser, Ronald F. "Loss Reserving." Foundations of Casualty Actuarial Science. Casualty Actuarial Society: 1990. pp. 143-229.